

# Quantum entanglement in random physical states

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Most states in the Hilbert space are maximally entangled. This fact has proven useful to investigate - among other things - the foundations of statistical mechanics. Unfortunately, most states in the Hilbert space of a quantum many body system are not physically accessible. We define physical ensembles of states by acting on random factorized states by a circuit of length  $k$  of random and independent unitaries with local support. This simulates an evolution for finite time  $k$  generated by a local (time-dependent) Hamiltonian. We apply group theoretic methods to study these statistical ensembles. In particular, we study the typicality of entanglement by means of the purity of the reduced state. We find that for a time  $k = O(1)$  the typical purity obeys the area law, while for a time  $k \sim O(L)$  the purity obeys a volume law, with  $L$  the linear size of the system. Moreover, we show that for large values of  $k$  the reduced state becomes very close to the completely mixed state.

*Introduction.*— Entanglement is the defining characteristic of quantum mechanics: it is a key ingredient in quantum information processing [1], quantum many-body theory [2], and the description of novel quantum phases of the matter [3]. More recently, quantum entanglement has shed new light on the foundations of statistical mechanics, and the processes of equilibration and thermalization. The idea is simple: even though quantum evolution keeps an isolated quantum system always in a pure state, the reduced state of a subsystem is mixed due to entanglement. One can then speculate whether the expectation values of local observables are typically close to those of the thermal state [4–6]. It has been shown that such typicality is related to the typicality of entanglement [6]. For instance, drawing a random state (with the Haar measure) from the Hilbert space of a quantum many-body system within some energy shell, its reduced state will be typically close to the Gibbs state at some temperature  $T$ . This is a kinematical argument: most states in the Hilbert space look locally thermal. With a similar reasoning, one can show that starting from a non thermal state the infinite time evolution for almost all initial states and almost all reasonable Hamiltonians will *almost always* produce a typical state in the Hilbert space, and then the kinematic argument follows. A series of beautiful and rigorous results has been found following this approach [6].

The problem with this approach is that typical states are not physical because they are not accessible in nature. What one can access are ground states or low energy states of local Hamiltonians, which are very little entangled because they obey an area law [7], and those that can be reached by means of evolution with a local Hamiltonian and for a reasonable amount of time, say, a time that scales at most polynomially in the size of the system. It is clear that the states that can be accessed in this way are a set of measure zero in the Hilbert space. One needs a doubly exponential time in the system size to access all the states of the Hilbert space, which is an unreasonable time scale for a macroscopic system. For this reason, some authors have argued that the Hilbert space is an illusion needed for mathematical consistency [8].

On the other hand, physical states do thermalize, as it has been shown in several experiments with cold atoms, theoret-

ical models and numerical simulations [9–11], or show typicality in the expectation value of local observables [12]. Does this mean that the mechanism for thermalization is not entanglement? Or the typical entanglement present in physical states is still enough to carry over the arguments about thermalization? There are several examples of physical relevance in which it is found that when the evolution time scales with the size of the system, the state is entangled with a volume law [13, 14]. But how typical is this? Can we prove any statement about the typicality of such situations?

In this letter, we want to answer the following question: how much are typical *physical* states entangled? To this end we define an ensemble  $\mathcal{E}^{(k)}$  of physical states in this way: pick a product state of a multi-partite system with local  $d$ -dimensional degrees of freedom, then act on this state with  $k$  independent random unitaries each of them compatible with some locality structure e.g., supported on edges of a graph. These stochastic circuits of local unitaries are, on the one hand, mimicking the continuous evolution generated over a time  $k$  by a local (possibly time-dependent) Hamiltonian; on the other hand, they are amenable to an elegant analytical treatment. In fact, by applying the group theoretic techniques of Ref [15] we are able to compute the ensemble average and variance of the purity of a sub region. The result is that, typically, for  $k = O(1)$  we obtain an area law, while for  $k$  scaling as the linear size  $L$  of the system the average purity shows a volume law. Moreover, we show that fluctuations are small, and that there is measure concentration around the average value. As a final result, we show that for  $k \rightarrow \infty$ , the subsystem typically reaches the completely mixed state.

*Statistical ensemble of physical states.*— We start by defining the ensembles  $\mathcal{E}^{(k)}$ ; henceforth we will refer to the elements of these ensembles as the *physical states*. Let  $V$  be a set of vertices endowed with a probability measure  $p : X \subset V \mapsto p(X) \in [0, 1]$  with  $\sum_X p(X) = 1$ . To each of the vertices  $x \in V$  we associate a local  $d$ -dimensional Hilbert space  $\mathcal{H}_x \simeq \mathbb{C}^d$ . The total Hilbert space is thus  $\mathcal{H}_V \equiv \otimes_{x \in V} \mathcal{H}_x$ , or the space of  $n$  *qudits*. A completely factorized state in  $\mathcal{H}_V$  has the form  $|\Phi\rangle = \otimes_{x \in V} |\phi_x\rangle$  and let  $\omega = |\Phi\rangle\langle\Phi|$  be its density matrix.

The statistical ensembles of quantum states  $\mathcal{E}^{(k)}$  are constructed in the following way: We first draw a subset  $X \subset V$  according to the measure  $p$  and then we draw a unitary  $U_X \in \mathcal{U}(\mathcal{H}_X)$  according to a chosen measure  $d\mu(U|X)$ . Then we define  $\mathcal{E}(p, d\mu) = \{U_X |\Phi\rangle\}_{X, U_X, \Phi}$ . This ensemble can be generalized to the  $k$ -iterated  $\mathcal{E}^{(k)}$  by considering unitaries of the form  $U = \prod_{i=1}^k U_{X_i}$  where the  $X_i$ 's ( $U_{X_i}$ 's) are drawn according the product probability  $\prod_{i=1}^k p(X_i) (\prod_{i=1}^k d\mu(U_{X_i}))$ : each tick of the clock (labelled by  $i$ ) a new independent set  $X_i$ 's and a unitary  $U_{X_i}$  are picked. In this ensemble, we can compute the statistical moments of any hermitian operator. It turns out that by varying over the  $U_X$  one already picks all the possible factorized  $\Phi$ 's so that the  $\Phi$  dependence can be in fact dropped.

*Subsystem purity.*— We want now inquire about typical entanglement in the statistical ensembles  $\mathcal{E}^{(k)}$ . Let us consider a bipartition  $V = A \cup B$  in the system:  $\mathcal{H}_V = \mathcal{H}_A \otimes \mathcal{H}_B$ , where  $\mathcal{H}_J = \otimes_{x \in J} \mathcal{H}_x$  with obvious notation. We take a state  $\rho \in \mathcal{E}_k$  and consider the reduced density matrix  $\rho_A = \text{Tr}_B(\rho)$ . In order to evaluate the entanglement of  $\rho$  we will compute the purity  $P = \text{Tr}_A(\rho_A^2)$ . To compute this trace we use the well-known fact that the trace over the square of every operator can be computed as the trace of two tensored copies of that operator times the swap operator. Indeed, defining  $\rho^{\otimes 2} = U^{\otimes 2} \omega^{\otimes 2} U^{\dagger \otimes 2}$  and considering the order two shift operator (swap) on  $T_x : \mathcal{H}_x^{\otimes 2} \mapsto \mathcal{H}_x^{\otimes 2}$ , we then have  $P = \text{Tr}_A(\rho_A^{\otimes 2} \tilde{T}_A) = \text{Tr}[\rho^{\otimes 2} T_A]$  where  $\tilde{T}_A = \otimes_{x \in A} T_x : (\mathcal{H}_A)^{\otimes 2} \mapsto (\mathcal{H}_A)^{\otimes 2}$  by  $|i_1, i_2\rangle \mapsto |i_2, i_1\rangle$  is the order 2 shift operator in the  $A$  space  $\mathcal{H}_A = \otimes_{i \in A} \mathcal{H}_i$  and  $T_A : (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes 2} \rightarrow (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes 2}$  is given by  $T_A = \tilde{T}_A \otimes \mathbb{1}_B$ . We can now consider different concrete ensembles. As a first basic model, let us consider the

*Single edge model.*— The system  $\mathcal{E}_{edge}$  consists of two sites  $A = \{i\}$  and  $B = \{j\}$  connected by an edge  $e$  so that the Hilbert space is  $\mathcal{H}_e = \mathcal{H}_A \otimes \mathcal{H}_B$ . The probability distribution is the trivial  $p(e) = 1$  and we pick the unitaries  $U_e(\mathcal{H}_e)$  with the Haar measure:  $d\mu(U_e) = d\mu_{Haar}$ . Following Ref.[15] one can exploit the group theoretic structure of the ensemble  $\mathcal{E}$  to compute average and statistical moments of operators. The average of an operator over a group action is indeed the weighted sum of projectors onto the IRreps of the representation of that group. The weights are the traces of the operator multiplied by the irrep projectors  $\Pi_j$  (divided by their dimensions  $d_j$ ). In our case the IRreps are carried by the totally symmetric ( $\mathcal{H}_+^2$ ) and totally antisymmetric ( $\mathcal{H}_-^2$ ) subspaces of  $\mathcal{H}_e^{\otimes 2}$ . The average is given by

$$\bar{P}^U = \int dU \text{Tr}[\rho^{\otimes 2} T_A] = \text{Tr}[\omega^{\otimes 2} \int dU (U^\dagger)^{\otimes 2} T_A U^{\otimes 2}]$$

Performing the integration over the unitaries we get  $\bar{P}^U = d_+^{-1} \text{Tr}(\Pi_+ T) \text{Tr}(\omega^2 \Pi_+) = d_+^{-1} \text{Tr}(\Pi_+ T)$  since  $\omega^{\otimes 2}$  is supported only in the totally symmetric subspace  $\Pi_+$  whose dimension is  $d_+ = d^2(d^2 + 1)/2$  and  $\text{Tr}[\omega^{\otimes 2} \Pi_+] = 1$ . The projector onto the totally symmetric space has the form  $\Pi_+ = (\mathbb{1} + T_i T_j)/2 \implies 1/2 \text{Tr}((\mathbb{1}_{(i,j)}^{\otimes 2} + T_i T_j) T_i) = d^3$  and

we finally get  $\bar{P}^U = d^3/d_+ = 2d/(d^2 + 1) \equiv 2N_d$ . In [15] it was defined the average entangling power  $\overline{e_p(U)}^U$  as the average entanglement one attains from a factorized bipartite state by averaging over the unitaries in the whole space with the Haar measure. With this definition,  $\overline{e_p(U)}^U := 1 - \bar{P}^U = 1 - 2N_d = (d - 1)^2/(d^2 + 1)$  [see Eq. (5) in [15]].

*Superoperator formulation.*— Going back to the general case, it is useful to define the superoperator that averages over the unitaries  $U_X \in \mathcal{U}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes 2})$   $\mathcal{R}_X(T_A) := \int d\mu(U|X) (U_X^\dagger)^{\otimes 2} T_A (U_X)^{\otimes 2}$ . Notice that when  $d\mu$  is the Haar measure the  $\mathcal{R}_X$  are projection superoperators; in the rest of the letter we will focus on this case. Then we can evaluate the average purity as

$$\bar{P} = \langle \omega^{\otimes 2}, \mathcal{R}(T_A) \rangle \quad (1)$$

where  $\mathcal{R} = \sum_{X \subset V} p(X) \mathcal{R}_X$  is a self-dual (hermitean) superoperator. As far as the purity calculations are concerned this superoperator completely characterizes the ensembles  $\mathcal{E}^{(k)}$ . Indeed, it is now easy to see that—in view of the statistical independence of each iteration—the average purity for the  $k$ -iterated ensemble  $\mathcal{E}^{(k)}$  is given by the expression (1) with  $\mathcal{R}$  replaced by  $\mathcal{R}^k$ . In order to understand the spectral properties of  $\mathcal{R}$  observe that:  $\|\mathcal{R}\| \leq \sum_{X \subset V} p(X) \|\mathcal{R}_X\| \leq \sum_{X \subset V} p(X) = 1$ . Since  $\mathcal{R}(\mathbb{1}) = \mathbb{1}$  we then see that  $\|\mathcal{R}\| = 1$  whence the eigenvalues  $\lambda_\alpha$  of  $\mathcal{R}$  are bounded in modulus by one and the highest one is  $\lambda_1 = 1$ . One can then write  $\bar{P}_k = \sum_\alpha \lambda_\alpha^k c_\alpha$ ; where  $c_\alpha := \langle \omega^{\otimes 2}, (T_A)_\alpha \rangle$  and  $(T_A)_\alpha$  denotes the projection of  $T_A$  onto the eigenvalue  $\lambda_\alpha$  eigenspace of  $\mathcal{R}$ . For  $k \rightarrow \infty$  this quantity goes to the limite value  $c_1$  while the convergence rate is dictated the second highest eigenvalue  $\lambda_2$  of  $\mathcal{R}$ :  $\bar{P}_k - \bar{P}_\infty \simeq c_2 \exp(-k \log 1/|\lambda_2|)$ , ( $k \rightarrow \infty$ ).

One of the key steps to obtain the results of this letter is to realize that the  $\mathcal{R}$ 's superoperators can be regarded as maps on the  $2^{|V|}$ -dimensional space spanned by the  $T_X$ 's ( $X \subset V$ ) into itself (instead of maps of the  $d^{4|V|}$ -dimensional  $\mathcal{L}(\mathcal{H}_V^{\otimes 2})$  into itself). For example, if  $X := \{a, b\}$  i.e., an edge and  $A$  is any subset of  $V$ , a calculation similar to the above leads to

$$\begin{aligned} \mathcal{R}_X(T_A) &= N_d(T_{A \setminus X} + T_{A \cup X}) \quad X \cap A \neq \emptyset \wedge X \cap B \neq \emptyset \\ \mathcal{R}_X(T_A) &= T_A \quad \text{otherwise} \end{aligned} \quad (2)$$

Moreover, the edge  $\mathcal{R}_X$ 's have low-dimensional invariant subspaces of permutations, e.g., in a chain topology the span of the  $T_A$ 's associated with connected  $A$ 's is invariant. This remark along with [16] allows for drastic simplifications in the evaluation of the average purity of  $\mathcal{E}^{(k)}$ .

To illustrate the usefulness of the algebra (2), let us compute again the purity for the single edge model by means of  $\mathcal{R}_X$ . Here,  $A$  is just one site and therefore  $A \setminus e = \emptyset$  and  $A \cup e = \{A, B\}$ . Moreover,  $T_A(\emptyset) = I$  and  $\mathcal{R}_e(T_A) = (I + T_A T_B)$ . Finally we get  $\bar{P} = \langle \omega^{\otimes 2}, T_A \rangle = 2N_d$  [16]. In the case of qubits,  $d = 2$  and  $2N_2 = .8$ . It is also possible to compute the variance by generalizing the group averages to higher power of the density operator to obtain  $\Delta P = .017$ . A systematic treatment of how to compute variances in these statistical ensembles is to be found in [17].

*Random edge model.*— At this point, we can implement some graph theoretic structure in the set  $V$  to make sense of the notion of locality for the unitaries  $U$ . Consider now the  $k$ -random edge model  $\mathcal{E}_{\text{random}}^{(k)}$  defined on a graph  $\Gamma = (V, E)$ . We define a flat probability distribution on the edges of the graph  $\Gamma$ :  $p(X) = 1/|E|$  if  $X \in E$  and zero otherwise. Then we pick the unitaries on the edges with the Haar measure:  $d\mu(U|X) = d\mu_{\text{Haar}}$ .

We call  $\partial A \subset E$  the subset of edges that have non null intersection with both  $A$  and  $B$ . The probability of an edge to belong to  $\partial A$  is thus  $q = |\partial A|/|E|$ . We are interested in thermodynamical situations where  $q \ll 1$ . Exploiting Eq. (2) we see that

$$\mathcal{R}(T_A) = \sum_{X \in E \setminus \partial A} p(X) T_A + \sum_{X \in \partial A} p(X) [T_{A \cup X} + T_{A \setminus X}] \quad (3)$$

where  $X \in E$  is an edge of the graph. For  $k = 1$  is then easy to find the average purity:  $\bar{P} = \sum_{X \in E \setminus \partial A} p(X) \langle \omega^{\otimes 2}, T_A \rangle + \sum_{X \in \partial A} N_d \langle \omega^{\otimes 2}, T_{A \cup X} + T_{A \setminus X} \rangle = 1 - (1 - 2N_d)q$ . From Eq.(3) we see that every application of  $\mathcal{R}_X$  transforms the subset  $A$  into a superposition of  $A \cup X$  and  $A \setminus X$  so that at any successive iteration the boundary of the new subset changes and its boundary length may change. To understand the structure of  $\mathcal{R}^k(T_A)$  it is instructive to consider explicitly the  $k = 2$  case. Iterating (3) and computing the purity using (1) [16] we find  $\bar{P} \simeq (1 - q)^2 + (1 - q)2qN_d + (2qN_d)^2 = (1 - q(1 - 2N_d))^2$  where we have used  $|\partial(A \cup e)| \simeq |\partial(A \setminus e)| \simeq |\partial A|$ . This is true e.g. if every vertex has degree  $o(|\partial A|)$ . The calculation easily extends to the case  $k = o(|\partial A|)$  and one finds  $\bar{P}_k \simeq (1 - q(1 - 2N_d))^k$ . Therefore the average 2-Renyi entropy of the ensemble is  $\bar{S}_2(k) \equiv -\log \bar{P}_k \geq -\log \bar{P}_k \simeq -k \log(1 - q(1 - 2N_d)) \simeq (1 - 2N_d)qk$ . In terms of the average entangling power one gets  $\bar{S}_2(k) \geq k|\partial A|/|E| \overline{e_p(U)}^U$ . In words: the average 2-Renyi entropy for the random edge model of the  $k$ -th iteration is lower bounded by  $k$  times the fraction of vertices in the boundary of the region  $A$  times the average entangling power of an edge unitary. This shows that there is a linear increase of entropy in time. Moreover, one can compute variances of  $\bar{P}$  and show that  $\Delta P \simeq |\partial A|$  [17] so that  $\sqrt{\Delta P}/\bar{P} \simeq 1/\sqrt{|\partial A|}$ . This in turn implies measure concentration (typicality) in the thermodynamic limit  $|\partial A| \rightarrow \infty$ . In the extreme case in which the graph is complete, then  $q = 1$  and one would expect that for  $k$  large the completely mixed state is reached [18].

This random edge model is of course ultra-local and then somewhat unphysical. The associated ensemble, roughly speaking, corresponds to the set of states obtainable from a product state by the action of a time dependent Hamiltonians supported on a *single* edge. We now move to consider a more physically satisfactory scenario.

*The linear chain.*— We now consider a model in which all the unitaries act on all the edges of the graph  $\Gamma$ . The probability distribution is thus  $p(X) = 1$  for  $X = V$  and zero otherwise. For the sake of simplicity in the following, we will consider the case of the graph  $\Gamma$  being a bipartite chain

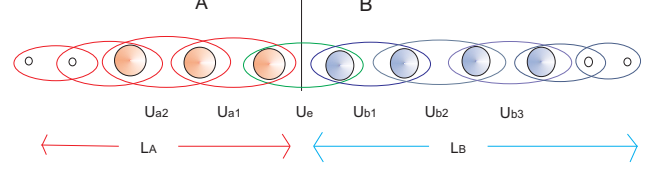


FIG. 1. A bipartite  $(A, B)$  spin chain of length  $L = L_A + L_B$  with nearest-neighbor qubits interacting via 2-qubit gates (ellipses). The edge  $e$  is the one that straddles the two partitions. The gates are numbered by the subscript  $xi$  where  $x = A, B$  denotes the two halves of the chain and  $i$  the distance from the boundary of the two partitions.

of length  $L = L_A + L_B$ . Extensions to higher dimensional geometries will be presented in [17]). We will label by  $U_e$  the unitary acting on the edge straddling the  $(A, B)$  bipartition, while we will use the labels  $a_i, b_i$  for the unitaries that act in the bulk of  $A, B$  respectively (see Fig.1). We label the sites of the chain as  $L_A, \dots, 1_A, 1_B, \dots, L_B$ . Since the unitary is a product of all the edge unitaries, we need to specify in which order they act. In the following, the unitary  $U_\sigma$  will always denote the product over all the edges in  $E$  with the order given by the permutation  $\sigma$ , so  $U_\sigma$  is the ordered product over local two-qudit unitaries. This corresponds to the (time ordered) infinitesimal evolution with a local Hamiltonian, where  $\sigma$  gives the time ordering:  $U_\sigma = U_{\sigma(e_1)} \dots U_{\sigma(e_{|E|})}$ . At this point we construct the set  $\mathcal{E}_{(\sigma)}^{(k)}(\Gamma) = \{U_\sigma |\Phi\rangle\}_U$  with measure  $d\mu(U) = \delta(U - U_\sigma) \prod_{e \in E} d\mu_{\text{Haar}}(U_e)$ . This ensemble approximates all the states that can be evolved from a factorized state with a local Hamiltonian acting for an infinitesimal amount of time. By  $k$ -iteration, we obtain the time evolution for a finite time  $k$ :  $\mathcal{E}_{\text{chain}}^{(k)}(\Gamma) = \{\prod_{i=1}^k U_{\sigma_i} |\Phi\rangle\}_{U, \sigma}$ . Here we consider all the possible ordered sequences of unitaries by taking, at each time step, a permutation  $\sigma$  of the edges uniformly at random.

This ensemble approximates all the states that can be reached in time  $k$  by the evolutions originated by all the possible time-dependent Hamiltonians on a graph. The ensemble  $\mathcal{E}^{(k)}$  thus only depends on the number of iterations (the "time")  $k$  and the graph  $\Gamma$ . The loss of purity due to the action of the unitaries depends on their order. Thus, in order to find an upper bound to the average purity, we consider the ordering that gives the minimum loss of purity. We can see that, for  $k = 1$ , acting in  $A$  and  $B$  after having acted on the edge  $e$  does not change the purity, so a sequence of the type  $U_A U_B U_e$  results in a minimal loss of purity. Moreover, the order of the unitaries inside  $A$  and  $B$  also counts. From the iteration of the algebra Eq.(2) we can see that if we pick the ordering in which we first act near the boundary and proceed towards the outer parts of the chain:  $U_A U_B U_e$  where  $U_A = U_{a_{L_A}} U_{a_{L_A-1}} \dots U_{a_2} U_{a_1}$  and  $U_B = U_{b_{L_B}} U_{b_{L_B-1}} \dots U_{b_2} U_{b_1}$

we will get the lowest possible powers of  $N_d$  and correspondingly the least decrease of purity. As  $k$  increases, the difference between different orderings is attenuated and for very large values of  $k$  it can also be neglected. We will anyway always consider the worst case scenario of ordering  $\sigma$  which corresponds to the minimal decrease of purity.

The action of the superoperator  $\mathcal{R}$  in the present chain case is more complicated because now  $X$  is not just the support of one unitary, but it contains the ordered product of all the edges. In particular, notice that now  $\mathcal{R}$  is not hermitean. Using Eq.(2) multiple times we find: for  $k = 1$ ,  $\mathcal{R}(T_A) = N_d(T_{A-1} + T_{A+1})$ . Where we used the notation  $A + r = A \cup \{1_B, \dots, r_B\}$  and  $A - r = A \setminus \{1_A, \dots, r_A\}$ . At the second iteration  $k = 2$ , we get

$$\mathcal{R}^2(T_A) = N_d^2 T_{A-2} + 2N_d^3 T_{A-1} + 2N_d^3 T_{A+1} + N_d^2 T_{A+2}.$$

We can see that nodes at distance 2 from the boundary enter the expression. Each  $T$  in the expression for  $\mathcal{R}^k(T_A)$  gives a 1 when we take the scalar product with  $\omega^{\otimes 2}$ . So we find  $\bar{P}_{k=2} = 2N_d^2 + 4N_d^3$ . It is important to understand how the interactions propagate with  $k$ . A somewhat lengthy calculation shows that as  $k$  increases, nodes at distance  $k$  from the edge participate to the averaging procedure and for every node that participates we pick a power for the base  $N_d$ . The result of the calculation is that  $\bar{P}_k = \sum_{m=0}^{k-1} 2 \binom{k+m-1}{m} N_d^{k+m}$ . This is valid for  $k < L_A$ . Summing the series for  $\bar{P}_k$  for large values of  $k$  one finds  $\bar{P}_k \simeq 2 [N_d / (1 - N_d)]^k$ . Recall that this equation, in view of the choice of the  $\sigma$  corresponding to the  $U_\sigma$  with the least entangling power, is an upper bound for the purity in  $\mathcal{E}_{chain}^{(k)}$ . The exponential decay of the purity in  $k$  is due to the fact that all the qudits at distance  $k$  from the edge are getting mixed. The average 2-Renyi entropy is  $\bar{S}_2 \geq -\log \bar{P}_k$ , therefore we have the bound

$$\bar{S}_2 \geq k \log \left( \frac{1 - N_d}{N_d} \right) - \log 2 \simeq k \log d - \log 2 \quad (4)$$

(Last approximation holds for large  $d$ ). Eq. (4) for  $k = O(L_A)$  implies a) a *volume law* for the entanglement scaling b) typicality: a nearly minimal value of the average of purity (in view of the Markov inequality), forces also the fluctuations around this average to be small. For  $k > |L_A|$  one has no longer a linear increase of entanglement with time but observes a saturation. This is the type of behavior that has been found in examples of entanglement dynamics after a quench using CFT techniques [13, 19] [see also the recent papers [20], closer to the spirit of the present one].

To study the limit of average purity for  $k \rightarrow \infty$  we first notice that the chain superoperator is a ( $\sigma$ -ordered) product of (non-commuting) projections  $\mathcal{R}_{chain} = \mathcal{R}_{\sigma(|E|)} \cdots \mathcal{R}_{\sigma(1)}$ . This implies  $\|\mathcal{R}_{chain}\| \leq \prod_{e \in E} \|\mathcal{R}_e\| \leq 1$ ; again this means that all the eigenvalues of  $\mathcal{R}_{chain}$  are smaller in modulus than 1 and therefore asymptotically just fixed points e.g.,  $\mathbb{1}, T_V$ , contribution to  $T_A$  survives. If now one *assumes* that the symmetric combination  $\mathbb{1} + T_V$  is the only relevant fixed point one finds  $\bar{P}_{k \rightarrow \infty} = (d^{2L-L_A} + d^{L+L_A})/d^L(d^L + 1)$ . We have

checked this result by numerical simulations [17] for the least and most entangling  $\sigma$ 's but we conjecture it to hold true for all orderings and besides the one-dimensional chain scenario. For large  $|V| = L$  one has  $\bar{P}_{k \rightarrow \infty} \simeq d^{-L_A} + d^{L_A-L}$  that in turn for  $L_A \leq L/2$  shows that the asymptotic purity differs from that of the totally mixed state  $\mathbb{1}_A/d^{L_A}$  for terms of order  $d^{-L_A}$ . Finally, if  $L_A \gg 1$ , this implies that the vast majority of the states in  $\mathcal{E}^{(k)}$ , once reduced to  $A$ , are close in  $L_1$ -norm to the maximally mixed state.

*Conclusions.*— In this letter we addressed the problem of typical entanglement of physical quantum states. This issue is crucial to assess the relevance of arguments based on entanglement typicality for generic random states in the foundations of statistical mechanics and equilibration dynamics. To this aim we have defined statistical ensembles of physical states  $\mathcal{E}^{(k)}$  by considering product states on a multi-partite systems and evolving them with  $k$  independent stochastic local gates. Ensemble averages can be computed by introducing suitable superoperators and using group-theoretic tools as in [15]. For one-dimensional geometries (chains of length  $L$ ) we proved that a) For a finite  $k = O(1)$  time, the entanglement measured by the decrease in purity shows an area law., b) As the time increases, correlations propagate and for  $k = O(L)$  the size of the system states display a volume law. We expect these results to hold true for higher dimensional regular graphs e.,g., 2D lattices [17]. We would like also to stress that even if in this Letter we have used purity to quantify the entanglement, this method extends in a straightforward way to general  $\alpha$ -Renyi entropy by natural modifications of superoperator  $\mathcal{R}$  and permutation  $T_A$  in Eq. (1) [17]. As we have seen, statistical fluctuations are small: the entanglement behaviour described above is typical and typicality of entanglement is at the root of the local thermalization results. At this point, we may speculate as whether our results implies a local thermalization result for physical states. This would be important both for the thermalization problem and for quantum information processing.

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